

# Pointwise Lower bounds on the Heat Kernels of Uniformly Elliptic Operators in Bounded Regions

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## Abstract

We obtain pointwise lower bounds for heat kernels of higher order differential operators with Dirichlet boundary conditions on bounded domains in  $\mathbf{R}^N$ . The bounds exhibit explicitly the nature of the spatial decay of the heat kernel close to the boundary. We make no smoothness assumptions on our operator coefficients which we assume only to be bounded and measurable.

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## 1 Introduction

Pointwise lower bounds on the heat kernels for higher order elliptic operators were first obtained by Davies [1]. In this paper we address the question of pointwise lower bounds on heat kernels generated by uniformly elliptic differential operators with Dirichlet boundary conditions on bounded regions of  $\mathbf{R}^N$ .

It is helpful to give an indicative though a non-rigorous formulation of the family of higher order operators that we focus on in this paper. The operator is defined more completely through its quadratic form. Given a bounded domain  $\Omega$  in  $\mathbf{R}^N$  we express the operator of order  $2m > N$  as :

$$Hf(x) := \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} (-1)^{|\alpha|} D^\alpha (a_{\alpha,\beta}(x) D^\beta f(x)) \quad (1.1)$$

where  $a_{\alpha,\beta}$  are complex bounded measurable functions.

The associated quadratic form  $Q$

$$Q(f) := \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} \int_{\Omega} a_{\alpha,\beta}(x) D^\beta f(x) \overline{D^\alpha f(x)} \quad (1.2)$$

defined with domain equal to the Sobolev space  $W_0^{m,2}(\Omega)$  will be assumed to satisfy the ellipticity condition with a strictly positive constant  $c$

$$c^{-1} \|(-\Delta)^{\frac{m}{2}} f\|_2^2 \leq Q(f) \leq c \|(-\Delta)^{\frac{m}{2}} f\|_2^2 \quad (1.3)$$

We define the spectral gap

$$\mu = \inf_{f \in C_c^\infty(\Omega)} \frac{Q(f)}{\|f\|_2^2}$$

Since we have made the assumption that  $\frac{N}{2m} < 1$ , it will be informative to track the dependency on this constraint by defining the quantity  $0 < \epsilon \leq 1 - \frac{N}{2m}$  and

$$\gamma := m(1 - \epsilon) - \frac{N}{2}$$

For a given point  $x$  in  $\Omega$  we define it's distance from the boundary  $\partial\Omega$

$$d_x := d(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y|$$

We define the function  $\tau(t)$  such that

$$\tau(t) := \begin{cases} \mu e^{-2\mu t} & t > \frac{1}{\mu} \\ \frac{1}{t} e^{-\mu t - 1} & t \leq \frac{1}{\mu} \end{cases} \quad (1.4)$$

Our main result is the following theorem :

**Theorem 1.1.** *If the heat kernels generated by the differential operator  $H$  satisfies the inequalities*

$$k(t, x, x) \leq c \left(1 - \frac{N+2\gamma}{2m}\right)^{-1} t^{-\frac{N+2\gamma}{2m}} d_x^{2\gamma} \quad \text{when } t < \frac{2}{\mu}$$

$$k(t, x, x) \leq c \left(1 - \frac{N+2\gamma}{2m}\right)^{-1} e^{-\mu t} d_x^{2\gamma} \quad \text{when } t \geq \frac{2}{\mu}$$

We define the function  $u_\epsilon(t, x)$  such that

$$u_\epsilon(t, x) := \frac{c}{\epsilon} d_x^{2m(1 - \frac{N}{2m} - \epsilon)} \tau\left(\frac{t}{2}\right)^{1-\epsilon} e^{-\mu t}$$

with  $c > 0$  such that

$$k(t, x, x) \leq u_\epsilon(t, x)$$

and define the function

$$\delta_\epsilon(t, x) := \frac{k(t, x, x)}{u_\epsilon(t, x)}$$

We summarise some of the key lemmas of Davies'[1] theory.

**Lemma 1.2** (Davies [1]). *Given  $0 < \alpha, s < 1$  and  $p$  defined by*

$$p + (1 - p)\alpha s = s \quad (1.5)$$

$$k(ts, x, x) < u_\epsilon(\alpha ts, x)^{1-p} u_\epsilon(t, x)^p \left( \frac{k(t, x, x)}{u_\epsilon(t, x)} \right)^p$$

**Definition 1.3.** *If  $\omega_x$  is the distribution such that  $\langle f, \omega_x \rangle = f(x)$  then we define  $G_t(x, x)$*

$$G_t(x, x) := \langle (tH + 1)^{-1} \omega_x, \omega_x \rangle = \left\langle \int_0^\infty e^{-(tH+1)s} ds \omega_x, \omega_x \right\rangle$$

**Lemma 1.4** (Davies [1]).

$$\frac{G_t(x, x)}{u_\epsilon(t, x)} < \int_0^1 \frac{u_\epsilon(\alpha t s, x)}{u_\epsilon(t, x)} \delta_\epsilon(t, x)^p ds + e^{-1} \delta_\epsilon(t, x) \quad (1.6)$$

**Lemma 1.5.**

$$G_t(x, x) = \sup_{g \in \text{Dom} Q} \left\{ \frac{|g(x)|^2}{tQ(g) + \|g\|_2^2} : g \neq 0 \right\}$$

*Proof.*

$$\begin{aligned} G_t(x, x) &= \langle (tH + 1)^{-1} \omega_x, \omega_x \rangle \\ &= \langle (tH + 1)^{-\frac{1}{2}} \omega_x, (tH + 1)^{-\frac{1}{2}} \omega_x \rangle \\ &= \sup_{f \in L^2(\Omega)} \{ |\langle (tH + 1)^{-\frac{1}{2}} \omega_x, f \rangle|^2 : \|f\|_2 = 1 \} \\ &= \sup_{f \in L^2(\Omega)} \{ |(tH + 1)^{-\frac{1}{2}} f(x)|^2 : \|f\|_2 = 1 \} \\ &= \sup_{g \in \text{Dom} Q} \left\{ \frac{|g(x)|^2}{\|(tH + 1)^{\frac{1}{2}} g\|^2} : g \neq 0 \right\} \\ &= \sup_{g \in \text{Dom} Q} \left\{ \frac{|g(x)|^2}{tQ(g) + \|g\|_2^2} : g \neq 0 \right\} \end{aligned}$$

□

## 2 Estimation with Test Functions

Let  $\psi$  be a function in  $C_c^\infty(\Omega)$  defined as :

$$\psi(u) := \begin{cases} 1 & u = 0 \\ 0 & |u| \geq 1 \end{cases}$$

and then define the test function  $g_{(x,r)}$  as

$$g_{(x,r)}(y) = \psi\left(\frac{y-x}{r}\right)$$

It is then clear to see that  $g_{(x,r)} \in \text{Dom}(Q)$  iff  $x \in \Omega$  and  $r \leq d_x$ .

By direct integration we can see that we have the two estimates

$$\begin{aligned} \|g_{(x,r)}\|_2^2 &\leq \int_{B(x,r)} d^N \mathbf{y} \leq cr^N \\ Q(g_{(x,r)}) &\leq \int_{B(x,r)} |\Delta^{\frac{m}{2}} g_{(x,r)}(y)|^2 d^N \mathbf{y} \leq cr^{N-2m} \end{aligned}$$

for some constant  $c > 0$ . As a consequence we have the estimate

$$G_t(x, x) \geq \frac{c}{tr^{N-2m} + r^N} \quad (2.7)$$

for some constant  $c > 0$ .

**Lemma 2.1.** *There is a constant  $c > 0$  such that when  $t \leq d_x^{2m}$  then*

$$G_t(x, x) \geq ct^{-\frac{N}{2m}} \quad (2.8)$$

*when  $t \geq d_x^{2m}$  then*

$$G_t(x, x) \geq ct^{-1}d_x^{2m-N} \quad (2.9)$$

*Proof.* When  $t \leq d_x^{2m}$  we substitute  $r = t^{\frac{1}{2m}}$  into inequality (2.7).

When  $t \geq d_x^{2m}$  we substitute  $r = d_x$  into inequality (2.7).  $\square$

### 3 Short Time asymptotics

In this section we assume that  $t < \frac{2}{\mu}$ .

**Lemma 3.1.** *When  $t \leq d_x^{2m}$  then  $k(t, x, x) > ct^{-\frac{N}{2m}}$  for some constant  $c > 0$*

*Proof.* By lemma 2.1 we have

$$\frac{G_t(x, x)}{u_\epsilon(t, x)} \geq c\epsilon \left( \frac{t}{d_x^{2m}} \right)^{1-\frac{N}{2m}-\epsilon} e^{\frac{\mu t}{2}(1-\epsilon)}$$

We begin our estimate with

$$\alpha \int_0^\alpha v(\theta, Ts, x) \delta_\theta^{\frac{s}{\alpha}} ds < \alpha \int_0^\infty v(\theta, Ts, x) \delta_\theta^s ds$$

recalling that  $\alpha < 1$  and  $\delta < 1$ .

Now substituting  $v$  we have

$$\alpha \int_0^\infty v(\theta, Ts, x) \delta_\theta^s ds = a\alpha c_\theta^2 \frac{d_x^{2m\theta-N}}{T^\theta} \int_0^\infty s^{-\theta} e^{-sTs(1-\frac{\theta}{2})} e^{s \ln \delta_\theta} ds$$

and evaluating this we have

$$\alpha \int_0^\infty v(\theta, Ts, x) \delta_\theta^s ds = a\alpha c_\theta^2 \frac{d_x^{2m\theta-N}}{T^\theta} \left[ \ln \left( \frac{e^{sT(1-\frac{\theta}{2})}}{\delta_\theta} \right) \right]^{\theta-1} \Gamma \left( 1 - \frac{\theta}{2} \right)$$

where  $a$  is some constant, hence we conclude

$$\frac{\alpha}{v(\theta, T, x)} \int_0^\alpha u(\theta, Ts, x) \delta_\theta^{\frac{s}{\alpha}} ds < e^{sT(1-\frac{\theta}{2})} \left[ \ln \left( \frac{e^{sT(1-\frac{\theta}{2})}}{\delta_\theta} \right) \right]^{\theta-1} \Gamma \left( 1 - \frac{\theta}{2} \right)$$

and with these estimates we conclude from the Green-Heat inequality

$$a \frac{e^\theta}{c_\theta^2} \left( \frac{d_x^{2m}}{T} \right)^{\frac{N}{2m}-\theta} e^{sT(1-\frac{\theta}{2})} < e^{sT(1-\frac{\theta}{2})} \left[ \ln \left( \frac{e^{sT(1-\frac{\theta}{2})}}{\delta_\theta} \right) \right]^{\theta-1} \Gamma \left( 1 - \frac{\theta}{2} \right) + e^{-1} \delta_\theta$$

then dividing by  $e^{sT(1-\frac{\theta}{2})}$  we have

$$\frac{e^\theta}{c_\theta^2} \left( \frac{d_x^{2m}}{T} \right)^{\frac{N}{2m}-\theta} < \left[ \ln \left( \frac{1}{\kappa} \right) \right]^{\theta-1} \Gamma \left( 1 - \frac{\theta}{2} \right) + e^{-1} \kappa$$

where  $\kappa = \frac{\delta}{e^{sT(2-\theta)}}$ . We now set  $\theta = \frac{N}{2m}$  and by applying lemma ?? we see that for some constant  $c$

$$c < \left[ \ln \left( \frac{1}{\kappa} \right) \right]^{\frac{N}{2m}-1}$$

and

$$c < \kappa$$

where  $c$  is some positive constant, this is

$$c < e^{-sT(1-\frac{N}{4m})} \delta_{\frac{N}{2m}}$$

and hence

$$k(T, x, x) > ce^{sT(1-\frac{N}{4m})} v\left(\frac{N}{2m}, T, x\right)$$

Making the substitution  $u\left(\frac{N}{2m}, T, x\right)$

$$k(T, x, x) > \frac{c}{T^{\frac{N}{2m}}}$$

□

### 3.1 When $T \geq d_x^{2m}$

The only difference to our analysis from the case  $T \leq d_x^{2m}$  is the lower bound on the Greens function, here it is

$$G_T(x) > c \frac{d_x^{2m}}{T}$$

and this leads to the different inequality

$$\begin{aligned} \frac{G_T(x)}{u(\theta, T, x)} &> \frac{e^\theta}{c_\theta^2} \frac{d_x^{2m-N}}{T} \frac{T^\theta}{d_x^{2m\theta-N}} e^{sT(1-\frac{\theta}{2})} \\ &= \frac{e^\theta}{c_\theta^2} \left( \frac{d_x^{2m}}{T} \right)^{1-\theta} e^{sT(1-\frac{\theta}{2})} \end{aligned}$$

With this the Green-Heat Inequality becomes

$$\frac{e^\theta}{c_\theta^2} \left( \frac{d_x^{2m}}{T} \right)^{1-\theta} e^{sT(1-\frac{\theta}{2})} < e^{sT(1-\frac{\theta}{2})} \left[ \ln \left( \frac{e^{sT(1-\frac{\theta}{2})}}{\delta_\theta} \right) \right]^{\theta-1} \Gamma \left( 1 - \frac{\theta}{2} \right) + e^{-1} \delta_\theta$$

and as before we let  $\kappa = \frac{\delta_\theta}{e^{sT(1-\frac{\theta}{2})}}$  and divide by  $e^{sT(1-\frac{\theta}{2})}$ , to get

$$\frac{a}{c_\theta^2} \left( \frac{d_x^{2m}}{T} \right)^{1-\theta} < \left[ \ln \left( \frac{1}{\kappa} \right) \right]^{\theta-1} \Gamma \left( 1 - \frac{\theta}{2} \right) + e^{-1} \kappa$$

We continue by applying lemma ?? to conclude

$$\frac{a}{c_\theta^2} \left( \frac{d_x^{2m}}{T} \right)^{1-\theta} < \left[ \ln \left( \frac{1}{\kappa} \right) \right]^{\theta-1}$$

for some constant  $a$  We next obtain from this

$$ac_\theta^{\frac{2}{1-\theta}} \left( \frac{d_x^{2m}}{T} \right)^{\frac{1-\theta}{\theta-1}} = ac_\theta^{\frac{2}{1-\theta}} \left( \frac{d_x^{2m}}{T} \right)^{-1} > \ln \left( \frac{1}{\kappa} \right)$$

and inverting the logarithm and substituting for  $\kappa$

$$ac_\theta' e^{sT(2-\theta)} e^{-\frac{T}{d_x^{2m}}} \leq \delta_\theta$$

and so

$$k(T, x, x) > ac_\theta c_\theta' \frac{d_x^{2m\theta-N}}{T} e^{-\frac{T}{d_x^{2m}}}$$

## 4 Long Time Asymptotics

We now consider the situation where  $T > \frac{1}{s}$  As before we begin with the inequalities

$$G_T(x) > c \frac{d_x^{2m-N}}{T}$$

and the upper bound on the heat kernel for  $T > \frac{1}{s}$  is

$$u(\theta, T, x) = c_\theta^2 s^{-\theta} d_x^{2m-N} e^{-sT}$$

therefore

$$\begin{aligned} \frac{G_T(x)}{u(\theta, T, x)} &> c_\theta^{-2} s^{-\theta} \left( \frac{d_x^{2m-N}}{T} \right) d_x^{2m} \frac{e^{sT}}{d_x^{2m-N}} \\ &= c_\theta^{-2} s^{-\theta} \frac{e^{sT}}{T} \end{aligned}$$

From section 6.1 we deduce from the Green-Heat inequality

$$\frac{G_T(x, x)}{u(\theta, T, x)} < \int_0^\infty \frac{u(\theta, \alpha T s, x)}{u(T, x)} (\delta_\theta(T, x))^p + e^{-1} \delta_\theta \quad (4.10)$$

It is important to note at this point that  $u(\theta, \alpha T s, x)$  is formulated differently for  $s > \frac{1}{\alpha s T}$  and  $s < \frac{1}{\alpha s T}$  and so we use the estimate

$$\frac{G_T(x, x)}{u(\theta, T, x)} < \int_0^\infty \frac{v(\theta, \alpha T s, x)}{u(T, x)} (\delta_\theta(T, x))^p + e^{-1} \delta_\theta$$

Giving us the Green-Heat inequality

$$c_\theta^{-2} s^{-\theta} \frac{e^{sT}}{T} \leq \frac{e^{sT}}{d_x^{2m}} \left( \frac{d_x^{2m}}{T} \right)^\theta \left[ \ln \left( \frac{e^{sT(1-\frac{\theta}{2})}}{\delta_\theta} \right) \right]^{\theta-1} \Gamma \left( 1 - \frac{\theta}{2} \right) + e^{-1} \delta_\theta$$

we then multiply by  $e^{sT(\frac{\theta}{2}-1)} d_x^{2m}$

$$c_\theta^{-2} s^{-\theta} e^{\frac{sT\theta}{2}} \frac{d_x^{2m}}{T} \leq e^{\frac{sT\theta}{2}} \left( \frac{d_x^{2m}}{T} \right)^\theta \left[ \ln \left( \frac{e^{sT(1-\frac{\theta}{2})}}{\delta_\theta} \right) \right]^{\theta-1} \Gamma \left( 1 - \frac{\theta}{2} \right) + e^{-1} d_x^{2m} e^{sT(\frac{\theta}{2}-1)} \delta_\theta$$

By observing that  $d_x$  is bounded above and that

$$\frac{d_x^{2m}}{T} < 1$$

we get by setting  $\kappa = \frac{\delta_\theta}{e^{sT(1-\frac{\theta}{2})}}$  and applying lemma ??

$$ac_\theta^{-2} s^{-\theta} \left( \frac{d_x^{2m}}{T} \right) \leq c \left[ \ln \frac{1}{\kappa} \right]^{\theta-1}$$

and then

$$ac_\theta^{\frac{2}{1-\theta}} \left( \frac{d_x^{2m}}{T} \right)^{\frac{1}{\theta-1}} \geq c \left[ \ln \frac{1}{\kappa} \right]$$

and solving we get our lower bound

$$k(T, x, x) > ac'_\theta u(\theta, T, x) \exp \left( - \left( \frac{T}{d_x^{2m}} \right)^{\frac{1}{1-\theta}} \right)$$

hence

$$k(T, x, x) > ac_\theta^2 c'_\theta d_x^{2m-N} e^{-sT} \exp \left( - \left( \frac{T}{d_x^{2m}} \right)^{\frac{1}{1-\theta}} \right)$$

The situation for higher order operators is very different than for the case of second order differential operators. The heat kernel may be written as

$$k(t, x, y) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \psi_n(x) \psi_n(y)$$

where  $\psi_n$  are the eigenfunctions and  $\lambda_n$  are the eigenvalues given in increasing order. In the case of second order operators the ground state  $\psi_0$  is positive. We therefore have the asymptotic behaviour

$$k(t, x, x) \sim e^{-\lambda_0 t} \psi(x)^2$$

for large time. However since the ground state for higher order operators is not necessarily positive this type of behaviour is not necessarily true.

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## References

- [1] E.B. Davies, *Pointwise Lower Bounds On The Heat Kernels of Higher Order Elliptic Operators* Math Proc Camb Phil Soc 125 1999 p105-111
- [2] D.W.Robinson A.F.M.ter Elst *Local Lower Bounds on Heat Kernels* Positivity 2(1998), 123–151.